Localization of random band matrices when $W \ll N^{1/4}$

Wednesday, 22 March 2023 9:36

45 minute talk at "Mathematical challenges in quantum physics" workshop, PCTS, Princeton, March 22, 2023
Ron Peled, Tel Aviv University, on Sabbatical at IAS and Princeton Univ.

Random band matrix model:

$$H = \begin{pmatrix} V_1 & -T_1^* & & & 0 \\ -T_1 & V_2 & -T_2^* & & \\ & -T_2 & V_3 & -T_3^* & \\ & & \ddots & \ddots & -T_{n-1}^* \\ 0 & & & -T_{n-1} & V_n \end{pmatrix}$$

Hermitian
$W \times W$ block matrix
Dimension $N = nW$
Bandwidth W

Wegner orbital model

indep. Gaussian entries up to Hermitian constraint

(V_j) independent $GUE(W)$ matrices
(T_j) independent Ginibre(W) matrices

indep. Gaussian entries

Density $\rho(H) = \frac{1}{Z_{n,W}} \exp\left(-W\left(\sum_j \|V_j\|_{HS}^2 + \sum_j \|T_j\|_{HS}^2\right)\right)$
with respect to Lebesgue measure $dV_1 \cdots dV_n \, dT_1 \cdots dT_{n-1}$
on complex $W \times W$ Hermitian matrices and complex $W \times W$ matrices.

Conjecture (Casati-Molinari-Izrailev 1990, Fyodorov-Mirlin 1991): H exhibits localization when $W \ll \sqrt{N}$.
H exhibits delocalization when $W \gg \sqrt{N}$.

Our result (Cipolloni-Peled-Schenker-Shapiro 2022):
Localization holds when $W \ll N^{1/4}$.

Prev., parallel and later results:

Localization: Schenker (2009) $W \ll N^{1/8}$
P.-Schenker-Sodin-Shamis (2017) $W \ll N^{1/7}$
In parallel to us, Smart-Chen (2022) $W \ll N^{1/4}$
Afterward, Goldstein (2022) $W \ll N^{1/2}$
M. & T. Shcherbina (series of works 2014-2022) crossover at $N^{1/2}$ for related questions

Poisson statistics proved only when W fixed, $N \to \infty$ (Brodie-Hislop 2020)

Delocalization: results by Bao, Bourgade, Erdős, Knowles, Yang, Yau, Yin (2011-2021)

State-of-the-art by Bourgade, Yang, Yao, Yin $W \gg N^{3/4}$.

Overview of proof:

For energy level $z \in \mathbb{R}$, let $G_z := (H - zI)^{-1}$.

Overview

For energy level $z \in \mathbb{R}$, let $G_z := (H - zI)^{-1}$ be the resolvent, or Green's fcn., of H.
Think of G_z as a $W \times W$ block matrix and let $G_z(x,y)$ be its (x,y) block $(1 \le x,y \le n)$.

Our result: $\forall z \in \mathbb{R}, \forall s \in (0,1)\ \exists C_s, c_s > 0$ s.t. $\forall x,y$

$$\mathbb{E}\left[\|G_z(x,y)\|^s\right] \le W^{C_s} \exp\left(-c_s \frac{|x-y|}{W^3}\right)$$

E.g., operator norm

This is known to imply localization when $n \ll W^3$ $(\Leftrightarrow N \ll W^4)$

For simplicity, fix $z=0$.
Write G_j for $G_{z=0}(1,j)$ and focus on result for G_n.

Basic approach (Schenker 2009, suggestion of Aizenman):

1) a-priori bound:

$$\mathbb{P}(\|G_n\| > t) \le \frac{\mathrm{poly}(W,n)}{t}, \quad t > 1.$$

2) Explicit formula:

$$G_n = T_1^{-1} T_1^* T_2^{-1} T_2^* \cdots T_{n-1}^* T_n^{-1}$$

where
$T_1 := V_1$, $T_j := V_j - T_{j-1} T_{j-1}^{-1} T_{j-1}^*$ for $2 \le j \le n$. (*)

3) The formula suggests that $\log\|G_n\|$ is like a sum of independent contributions and hence should have wide spread. However, the a-priori bound shows $\log\|G_n\|$ is unlikely to be large. Hence $\log\|G_n\|$ should typically be small.

We will formalize this to prove

$$\mathbb{P}\left(\|G_n\| \ge e^{-c\frac{n}{W^3}}\right) \le C e^{-c\frac{n}{W^3}}.$$

Together with the a-priori bound, this will imply that for $s \in (0,1)$, $\mathbb{E}(\|G_n\|^s) \le W^{C_s} e^{-c_s \frac{n}{W^3}}$.

(1) and (2) are standard. We explain here our formalization of (3).

Details from proof:

Details from proof:

1) Change of variables: The (T_j) are Hermitian by (*).
We change variables from (V,T) to $(\mathcal{T},T)$. By (*) the Jacobian of this transformation is 1. Hence the density of $(\mathcal{T},T)$ is

$$\rho(\mathcal{T},T)=\frac{1}{Z_{n,W}}\exp\left(-W\left(\sum_j\|\mathcal{T}_j+T_{j-1}\mathcal{T}_{j-1}^{-1}T_{j-1}^{*}\|_{HS}^2+\sum_j\|T_j\|_{HS}^2\right)\right)$$

where we put $T_{j-1}\mathcal{T}_{j-1}^{-1}T_{j-1}^{*}=0$ for $j=1$.

2) Inducing fluctuations (an "adaptive Mermin-Wagner" transformation)

We seek to perturb $(\mathcal{T},T)$ in a way that doesn't change the density ρ too much but does change our target observable $\|G_n\|$ significantly.

Lemma (with origins in Pfister 1981, Richthammer 2007, Miłoś-Peled 2015):
Let X be a random variable in $\mathbb{R}^m$ with density ρ.
Let $S^+,S^-:\mathbb{R}^m\to\mathbb{R}^m$ be absolutely-continuous bijections.
Then $\forall$ event E

$$\sqrt{\mathbb{P}(X\in S^+(E))\,\mathbb{P}(X\in S^-(E))}\geq\alpha(E,S^+,S^-)\,\mathbb{P}(X\in E)$$

with

$$\alpha(E,S^+,S^-)=\inf_{x\in E}\frac{\sqrt{\rho(S^+(x))\,\rho(S^-(x))}}{\rho(x)}\sqrt{J^+(x)J^-(x)}$$

where $J^{\pm}:\mathbb{R}^m\to\mathbb{R}$ are the Jacobians

$$J^{\pm}(x):=\left|\det\left(\frac{\partial S^{\pm}}{\partial x_i}(x)\right)\right|$$

(so that $\mathbb{P}(X\in S^{\pm}(E))=\int_{S^{\pm}(E)}\rho(x)\,dx=\int_E\rho(S^{\pm}(x))J^{\pm}(x)\,dx$)

Proof: Set $I:=\int_E\sqrt{\rho(S^+(x))\,\rho(S^-(x))\,J^+(x)J^-(x)}\,dx$.

Then, on the one hand,

$$I \geq \alpha(E, S^+, S^-) \int_E \rho(x)dx = \alpha(E, S^+, S^-)\mathbb{P}(X \in E).$$

On the other hand, by C-S,

$$I \leq \left(\int_E \rho(S^+(x))J^+(x)dx \int_E \rho(S^-(x))J^-(x)dx\right)^{1/2} =$$

$$= \sqrt{\mathbb{P}(X \in S^+(E))\,\mathbb{P}(X \in S^-(E))}.$$

3) How to use the above lemma:

We define the mappings $S^{\pm}(\Gamma, T) = (\Gamma, T^{\pm})$

With $T_j^{\pm} = e^{\pm \delta F_j} T_j$ With a small $\delta > 0$

and $F_j = F_j(\|T_j\|_{HS}, \|V_{j+1}\|_{HS}, \|T_{j+1}\|_{HS}, \|T_j^{-1}\|_{HS}) \in [0,1]$

is a smooth fcn. which typically equals 1 but smoothly transitions to 0 if any of its arguments is abnormally large ($\|T_j\|_{HS} >> \sqrt{W}$, $\underline{\|V_{j+1}\|_{HS}}$, $\|T_{j+1}\|_{HS}$, $\|T_j^{-1}\|_{HS} >> W$)

These are bijections when $\delta W << 1$. (typically, $\|V_{j+1}\|_{HS}$ does not exceed $\sqrt{W}$ but this relaxed bound suffices for us)

It is proved that

$$\alpha(\text{full space}, S^+, S^-) \geq e^{-c_0 \delta^2 n W^3}. \quad (**)$$

Let

$$E = \{\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}}\} \cap \{\sum_j F_j \geq \tfrac{n}{2}\}$$

By lemma,

$$\sqrt{\mathbb{P}((\Gamma,T) \in S^+(E))} \geq \sqrt{\mathbb{P}((\Gamma,T) \in S^+(E))\mathbb{P}((\Gamma,T) \in S^-(E))} \geq e^{-c_0\delta^2 n W^3}\mathbb{P}(E)$$

i) $\mathbb{P}(E) \geq \mathbb{P}(\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}}) - \underbrace{\mathbb{P}(\text{at least half of the } F_j \text{ are not } 1)}_{\leq e^{-cn}}$

$\leq e^{-cn}$. Uses result of Aizenman-P.-Schenker-Shamis-Sodin (2017) to control norm of T_j^{-1} (inverse of GUE + arbitrary Hermitia is under control)

ii) $\mathbb{P}((\Gamma,T) \in S^+(E)) \leq \mathbb{P}(\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}} e^{\frac{1}{2}\delta n}) \leq \overset{\text{a-priori bound}}{\mathrm{Poly}(W,n)}\, e^{-\frac{1}{2}\delta n + \frac{c_0}{64}\frac{n}{W^3}}.$

Thus, $\mathbb{P}(\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}}) \leq \mathrm{Poly}(W,n)e^{-\frac{1}{4}\delta n + \frac{c_0}{128}\frac{n}{W^3} + c_0\delta^2 n W^3} + e^{-cn}$

Thus, $\mathbb{P}(\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}}) \leq \mathrm{Poly}(W,n)\, e^{-\frac{1}{4}\delta n + \frac{c_0}{128}\frac{n}{W^3} + c_0 \delta^2 n W^3} + e^{-cn}$

Choose $\delta = \frac{1}{8 c_0 W^3}$ to get $\mathbb{P}(\|G_n\| \geq e^{-\frac{c_0}{64}\frac{n}{W^3}}) \leq \mathrm{poly}(W,n)\, e^{-\frac{c_0}{128}\frac{n}{W^3}} + e^{-cn}$

as we wanted to prove.

4) The bottleneck for (**):

Examine $\inf_{(T,T)\in E} \frac{\sqrt{\rho(S^+(T,T))\,\rho(S^-(T,T))}}{\rho(T,T)}$

To estimate it, use that $\forall$ matrices A, B, C

$$\frac{1}{2}\left(\|A+B+C\|_{HS}^2 + \|A-B+C\|_{HS}^2\right) - \|A\|_{HS}^2 = \|B\|_{HS}^2 + 2\,\mathrm{Re}(\mathrm{tr}(C^* A))$$

The contributing term in the density ρ is

$$\|T_j + T_{j-1} T_{j-1}^{-1} T_{j-1}^* \|_{HS}^2$$

We have

$$T_j + T_{j-1}^{\pm} T_{j-1}^{-1} T_{j-1}^{\pm *} = T_j + T_{j-1} T_{j-1}^{-1} T_{j-1}^* e^{\pm 2\delta F_{j-1}}$$

$$= \underbrace{T_j + T_{j-1} T_{j-1}^{-1} T_{j-1}^*}_{=: A = V_j} + T_{j-1} T_{j-1}^{-1} T_{j-1}^* (e^{\pm 2\delta F_{j-1}} - 1)$$

So that $B \approx \delta F_{j-1} T_{j-1} T_{j-1}^{-1} T_{j-1}^* = \delta F_{j-1}(V_j - T_j)$

$C \approx \delta^2 F_{j-1}^2 T_{j-1} T_{j-1}^{-1} T_{j-1}^* = \delta^2 F_{j-1}(V_j - T_j)$

and we have $\|B\|_{HS}^2 \lesssim \delta^2 W^2$ ← This is the main contributing factor

$\mathrm{Re}(\mathrm{Tr}(C^* A)) \lesssim \delta^2 W^2$ ← This estimate can be reduced with tighter control on $\|V_j\|_{HS}$

as $F_{j-1}(\|V_j\|_{HS} + \|T_j\|_{HS}) \lesssim W$.

Recalling the extra factor of W multiplying the norms in the density ρ we get a factor of $\delta^2 n W^3$ in (**).